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Universality of correlations of levels with discrete statistics

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Abstract

We study the statistics of a system of N random levels with integer values, in the presence of a logarithmic repulsive potential of Dyson type. This probleme arises in sums over representations (Young tableaux) of GL(N) in various matrix problems and in the study of statistics of partitions for the permutation group. The model is generalized to include an external source and its correlators are found in closed form for any N. We reproduce the density of levels in the large N and double scaling limits and the universal correlation functions in Dyson's short-distance scaling limit. We also study the statistics of small levels.

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1 Introduction and definition of the generalized model

The theory of random matrices leads often to consider character expansions, i.e. sums over irreducible representations of groups such as $GL(N)$ or $U(N)$ [1, 2, 3, 4, 5, 6, 7, 8, 9]. Similar sums play also a role in probability theory [10], in particular in recent studies of the distribution of cycles in the group of permutations [11, 12, 13, 14, 15]. This has led to the study of probability distribution functions (PDF) for N non-negative integers $h_1, \cdots, h_N$ of the form :

$$\mathcal{P}_\alpha(h_1, \cdots, h_N) \sim \Delta^2(h)\alpha^{\sum_{k=1}^N h_k} \tag{1}$$

where $\Delta(h) = \prod_{m>j}(h_m - h_j)$, and $0 < \alpha < 1$ is a real parameter.

Let us generalize this distribution to an ensemble characterized by N parameters $\alpha_1, \cdots, \alpha_N$, in the following way :

$$\mathcal{P}_{[\alpha_1, \cdots, \alpha_N]}(h_1, \cdots, h_N) = \frac{1}{Z}\Delta(h)\chi_h(\{\alpha\}) \tag{2}$$

where $\chi_h(\alpha) = \dfrac{\det_{k,j} \alpha_k^{h_j}}{\Delta(\alpha)}$ is the Weyl character of a diagonal $GL(N)$ group element $\alpha_1, \cdots, \alpha_N$ of a given irreducible representation R. R is fixed in terms of the highest weights $m_k = h_k + k - N, \quad k = 1, \cdots, N$. (In principle the corresponding h_k's are strictly ordered, but in view of the symmetry of the weight (1) this restriction may be ignored in the sums). The constant Z is defined by the normalization condition

$$Z = \sum_{h_1=0}^{\infty} \cdots \sum_{h_N=0}^{\infty} \Delta(h)\chi_h(\{\alpha\}) \tag{3}$$

Note that in the limit of coinciding α's $\alpha_k \to \alpha$ the (non-normalized) distribution (2) reduces to (1); this follows from the well known formula:

$$\chi_h(\alpha) \to \prod_{k=1}^{N} \alpha^{\sum_k m_k} \dim_{\{m\}} = \frac{\Delta(h)}{\prod_{k=0}^{N-1} k!}\alpha^{\sum_k (h_k - k + N)} \tag{4}$$

where $\dim_{\{m\}}$ is the dimension of a representation given by the highest weights $m_1, \cdots, m_N$.

There are several reasons for this generalization. First it leads to simple exact formulae, as in the case of random matrices coupled to an external matrix source [16, 17]; the N parameters α_k's play here the role of the eigenvalues of the source. Here also, even when the source is a simple multiple of the identity, meaning now that all the α_k's are equal to a single α, the final formulae are explicit and simple. Furthermore these parameters α_k's provide a powerful check of universality. Indeed it is found here that at generic points, at which the density of levels is non-singular, the correlations are, in the proper Dyson scaling limit, insensitive to the specific probability distribution. Singular points fall also into universal classes, as for the usual matrix models [16]. Varying the external parameters α_k one can indeed tune various singular classes : examples are given in the subsequent sections. The machinery developed for integrable systems, and used in [11, 12, 13, 14, 15] for the single-parameter model (1), ought to be be applicable to our generalized model (2) as well.

Finally the distribution probability (1) provides a natural measure on the S_∞ group of arbitrary permutations ; the m_k's with $k = 1, 2, \cdots, N$, are the lengths of the cycles of a permutation class consisting exactly of N cycles. One can also interpret this distribution probability as defined on the (infinite) set of all Young tableaux with m_k boxes in the k'th row.

In our case, (2) is a natural multi-parametric generalization of (1), which may now be interpreted as a specific coloring of Young tableaux. The boxes of the Young tableau characterizing a permutation, are colored in N colors in a way which is explained below ; the k-th color is weighted with α_k. For instance, for $N = 2$, there are two rows of lengths (highest weights) $m_1 \geq m_2$ in the corresponding Young tableau characterizing a class of permutations with 2 cycles, and for the character we have the following finite sum of positive terms:

$$\chi_{m_1,m_2}(\alpha_1, \alpha_2) = \sum_{k=0}^{m_1-m_2} \alpha_1^{m_1-k} \alpha_2^{m_2+k} = \\ = \alpha_1^{m_1}\alpha_2^{m_2} + \alpha_1^{m_1-1}\alpha_2^{m_2+1} + \cdots + \alpha_1^{m_2+1}\alpha_2^{m_1-1} + \alpha_1^{m_2}\alpha_2^{m_1} \qquad (5)$$

This may be interpreted as a coloring of a tableau, in which the first $m_1 - k$ boxes

of the upper row have a color 1 and all the other boxes have color 2 ; we sum over k with a factor $\alpha_1^{\#1}\alpha_2^{\#2}$ where $\#1, \#2$ are the numbers of boxes ("areas") of the Young tableau of a given color.

To generalize this formula to any N, we have to expand a general character into a sum of monomials in the α's (generators of the maximal torus of $GL(N)$):

$$\chi_{\{m\}} = \sum_{\{l\}_{\{m\}}} n\{l\} \prod_{k=1}^{N} (\alpha_k)^{l_k} \tag{6}$$

where the finite sum over the positive integers $(l_1, \cdots, l_N)$ characterizing the elements of the Lie algebra of the maximal torus, or the weights of representation R, is restricted in a specific way by the shape of the corresponding Young tableau of the representation $R(m_1, \cdots, m_N)$. The positive integers $n\{l\}$ are called the multiplicities of those weights.

To make all this more explicit, and to give it a nice probabilistic interpretation, let us introduce the Gelfand-Tseytlin scheme (GT scheme), which provides an orthonormal basis of states for a given representation R of $GL(N)$ characterized by the highest weights $m_i,\ i = 1, \cdots, N$. Every state vector of R is characterized by $N(N-1)/2$ positive integers $m_{kj}, k = 1, \cdots, N,\ j = 1, \cdots, k$. The first ones, the $m_{N,i} \equiv m_i,\ i = 1, \cdots, N$, are simply the highest weights. The subsequent ones are given integers, restricted by the inequalities:

$$m_{j,i} \geq m_{j-1,i} \geq m_{j,i+1}, \quad i = 1, \cdots, j \tag{7}$$

The basis vectors of this representation are thus characterized by a triangular array usually depicted as [18]

$$\begin{array}{ccccccc} m_{N,1} & & m_{N,2} & & \cdots & & m_{N,N} \\ & m_{N-1,1} & & \cdots & & m_{N-1,N-1} & \\ & & \vdots & & \vdots & & \\ & & & m_{N,N} & & & \end{array} \tag{8}$$

in which any m in a given row lies in-between the two m's next to it in the previous row. We will use the definition of a character as a trace of a group element in its diagonal form, for a given representation R: $\chi_R(\{\alpha\}) = \mathrm{Tr_R} \prod_{\mathrm{j=1}}^{\mathrm{N}} \alpha_{\mathrm{j}}^{\mathrm{T^R_{jj}}}$, where T^R_{jj} is a diagonal generator in the Lie algebra of this representation. In the GT basis the values of T^R_{jj} are given as (see for example [18])

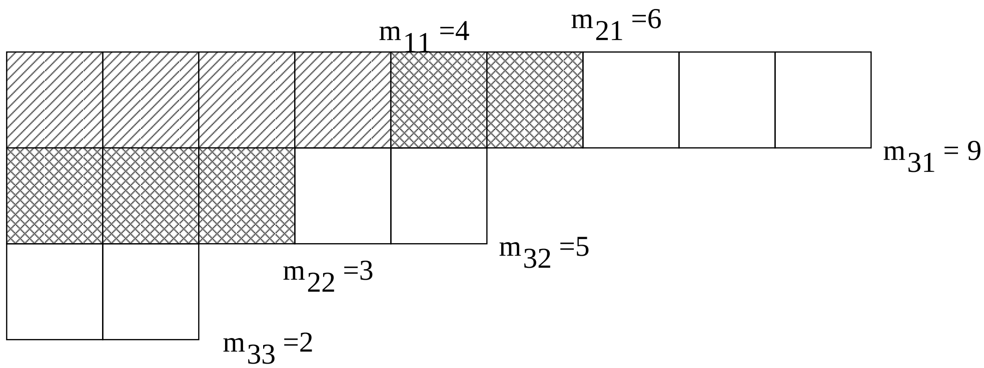
$$T^R_{jj} \equiv l_i = \sum_{i=1}^{j} m_{ji} - \sum_{i=1}^{j-1} m_{j-1,i} \tag{9}$$

The expansion (6) is now explicitely given by the formula:

$$\chi_R(\{\alpha\}) = \sum_{\{GT\}_R} \prod_{j=1}^{N} \alpha_j^{l_j\{m\}} \tag{10}$$

where the sum is taken over all GT schemes (states), satisfying the restrictions (7). Of course in this sum the same monomial may appear several times. The number of times each monomial enters in the sum is the multiplicity $n\{l\}$ of eq. (6).

This formula may be given an interesting probabilistic geometrical interpretation, in terms of sums over the colorings of Young tableaux. Let us define a coloring in the following way (see figure 1 for an illustration of the $N = 3$ case). The color number one is given to the last $m_{N,1} - m_{N-1,1}$ boxes of the first row, the last $m_{N,1} - m_{N-1,1}$ boxes of the second row, etc., the $m_{N,N}$ boxes of the last row. Color number 2 is given to the boxes numbered $m_{N-1,1} + 1$ to $m_{N-2,1}$ in the first row, $m_{N-1,2} + 1$ to $m_{N-2,2}$ in the second row, etc., the first $m_{N-1,N-1}$ boxes of the row before last ; etc., the N-th color is given to the first $m_{1,1}$ boxes of the first row. At the end, the Young tableau is a "brick wall" made out of colored bricks, each brick at an upper level relying on zero, one or two lower bricks of lower color indices, as depicted on fig.1.



$l_1 = m_{11}$ = the area covered by

$l_2 = m_{21} + m_{22} - m_{11}$ = the area covered by

$l_3 = m_{31} + m_{32} + m_{33} - m_{21} - m_{22}$ = the area covered by

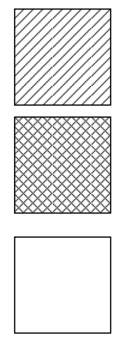


Fig. 1: Coloring of the Young tableau for the term in the distribution probability proportional to $\alpha_1^{l_1} \alpha_2^{l_2} \alpha_3^{l_3}$.

Note that if all the α_k's reduce to $\alpha_k = 1$, one is simply summing in (10) over each state with weight one ; therefore one recovers the dimension of the representation dim_R as stated in (4).

If one substitutes the expression (10) for the character into (2), one obtains a probability for the colored Young tableaux (related to permutations of colored objects).

In the next sections, we derive explicit (for any N) formulae for the correlators of the distribution (2) and study various scaling limits.

2 Density of discrete levels

The density of levels is defined as

$$\rho(\sigma) = \langle \frac{1}{N}\sum_k \delta_{N\sigma,h_k}\rangle = \oint \frac{dt}{2i\pi t^{N\sigma+1}}\langle \frac{1}{N}\sum_k t^{h_k}\rangle \ , \tag{11}$$

in which the brackets denote the averaging with respect to the probability distribution (2). The variable $N\sigma$ is a priori an integer but, in the large N-limit, σ will be considered as a continuous variable. Let us calculate $U_1(t) \equiv \langle \frac{1}{N}\sum_k t^{h_k}\rangle_h$ by performing explicitly the sums over the h's. Using the antisymmetry of $\Delta(h)$ in (2) we can rewrite it as

$$U_1(t) = C\sum_k\sum_h t^{h_k}\frac{\det_{m,j}\alpha_m^{h_j}}{\Delta(\alpha)} = \frac{C}{\Delta(\alpha)}\sum_k\sum_P(-)^P\sum_h\prod_m\left(\alpha_m^{(k)}\right)^{h_m} h_m^{P_m-1} \tag{12}$$

where the determinant is represented as a sum over permutations P of the levels $h_1,\cdots,h_N$ and by definition

$$\alpha_m^{(k)} = t^{\delta_{m,k}}\alpha_m. \tag{13}$$

The sums over h's are now independent and can be calculated using the formula :

$$\sum_{h=0}^{\infty} h^p\alpha^h = (\alpha\partial_\alpha)^p\frac{1}{1-\alpha} = \frac{1}{p!}\ \frac{1}{1-\alpha}Q_p\left(\frac{1}{1-\alpha}\right)$$

where $Q_p(x)$ is a polynomial of degree p, whose coefficient of highest degree is one. Since for any set of such polynomials $\det_{p,q} Q_{p-1}(x_q) = \Delta(x_1,\cdots,x_N)$ we obtain from (12) :

$$U_1(t) = \frac{C}{\Delta(\alpha)}\sum_k\frac{\Delta\left((1-\alpha^{(k)})^{-1}\right)}{\prod_m\left(1-\alpha_m^{(k)}\right)} = \frac{1}{N}\sum_k\left[\frac{1-\alpha_k}{1-t\alpha_k}\right]^N {\prod_j}' \ \frac{t\alpha_k-\alpha_j}{\alpha_k-\alpha_j} \tag{14}$$

where prime in the product means that the term $k=j$ is omitted. The constant C has been determined, through the normalization condition $U_1(1) = 1$.

This quantity has an elegant contour integral representation, similar to the one found in [16] for integrals over hermitian random matrices with an external matrix source. Indeed, the integral

$$U_1(t) = \frac{1}{N(t-1)}\oint\frac{du}{2i\pi u}\left(\frac{1-u}{1-tu}\right)^N\prod_j\frac{tu-\alpha_j}{u-\alpha_j}, \tag{15}$$

over a contour in the complex u-plane which surrounds the N poles α_j's (and not the origine), reproduces the sum (14). For the density of levels (11), since

$$\rho(\sigma) = \oint \frac{dt}{2i\pi} \frac{U_1(t)}{t^{N\sigma+1}}, \tag{16}$$

where the integration contour encircles the vicinity of the origin, we obtain, using (15):

$$\rho(\sigma) = \frac{1}{N} \oint \frac{dt}{2i\pi t^{N\sigma+1}(t-1)} \oint \frac{du}{2i\pi u} \left(\frac{1-u}{1-tu}\right)^N \prod_j \frac{tu-\alpha_j}{u-\alpha_j}. \tag{17}$$

Similarly for the resolvent, defined as $G(z) = \langle \frac{1}{N} \sum_{k=1}^{N} \frac{1}{z - h_k/N} \rangle$, we can use

$$G(z) = \int_0^\infty d\tau e^{-\tau z} \; U_1(e^{\tau/N}) \tag{18}$$

and obtain the integral representation:

$$G(z) = \frac{1}{N} \int_0^\infty \frac{d\tau}{(e^{\tau/N}-1)} e^{-\tau z} \oint \frac{du}{2i\pi u} \left(\frac{1-u}{1-e^{\tau/N}u}\right)^N \prod_j \frac{e^{\tau/N}u-\alpha_j}{u-\alpha_j} \tag{19}$$

After the change $t \to t/u$ in (17) we get ($\sigma = p/N$):

$$\rho(p/N) = \frac{1}{N} \oint \frac{dt}{2i\pi t^{p+1}} \prod_j \frac{(t-\alpha_j)}{(t-1)} \oint \frac{du u^{p+1}}{2i\pi} \prod_j \frac{(u-1)}{(u-\alpha_j)} \frac{1}{t-u} \tag{20}$$

or, inflating the contour of integration in u and changing $u \to 1/w$,:

$$\rho(p/N) = \frac{1}{N} \oint \frac{dt}{2\pi t^{p+1}} \prod_j \frac{(\alpha_j - t)}{(1-t)} \oint \frac{dw}{2\pi w^{p+2}} \prod_j \frac{(1-w)}{(1-\alpha_j w)} \frac{1}{tw-1} \tag{21}$$

Expanding the last factor we get a finite sum representation for the correlator:

$$\rho(p) = \frac{-1}{N} \sum_{k=0}^{p} \mathcal{L}_N^{p-k}(\alpha) \hat{\mathcal{L}}_N^{p-k+1}(\alpha) \tag{22}$$

where the polynomials in α's $\mathcal{L}_N^q(\alpha)$ and $\hat{\mathcal{L}}_N^q(\alpha)$ are defined as

$$\mathcal{L}_N^q(\{\alpha\}) = \oint \frac{dx}{2\pi i x^{q+1}} \prod_j \frac{(\alpha_j - x)}{(1-x)} \tag{23}$$

$$\hat{\mathcal{L}}_N^q(\{\alpha\}) = \oint \frac{dx}{2\pi i x^{q+1}} \prod_j \frac{(1-x)}{(1-\alpha_j x)} \tag{24}$$

The contours for both (23, 24) encircle the pole at $x = 0$.

Note that this expression is exact, for any N and any set of α's. In particular for the density, in the simplest case $\alpha_1 = \alpha_2 = \cdots = \alpha_N = \alpha$, most studied in the literature, this leads to a simple integral:

$$\rho(p/N) = \frac{-1}{N} \oint \frac{dt}{2\pi t^{p+1}} \frac{(t-\alpha)^N}{(t-1)^N} \oint \frac{du u^{p+1}}{2\pi} \frac{(u-1)^N}{(u-\alpha)^N} \frac{1}{t-u} \tag{25}$$

or, after inflating the contour and changing $u \to \alpha/w$

$$\rho(p/N) = -\oint \frac{du}{2i\pi u} \oint \frac{dt}{2i\pi t^{N\sigma+1}(t-1)} \left(\frac{(1-u)(\alpha - tu)}{(1-tu)(u-\alpha)} \right)^N . \tag{26}$$

The polynomial (23) reduces now to:

$$\mathcal{L}_N^q(\alpha) = \oint \frac{dx}{2\pi i x^{q+1}} \left(\frac{\alpha - x}{1-x} \right)^N \tag{27}$$

and the two sets of polynomials are related here simply by

$$\mathcal{L}_N^q(\alpha) = \alpha^{N-q} \hat{\mathcal{L}}_N^q(\alpha). \tag{28}$$

In the next sections we will study the large N limit of this density and, for the case of equal α's, compare it with a direct computation based on the solution of a Riemann-Hilbert problem given in the appendix.

3 Pair correlator of discrete levels

The simplest interesting correlator, which is conjectured to obey a short distance universality for nearby levels is the pair correlator:

$$U_2(t_1, t_2) \equiv \frac{1}{N^2} \langle \sum_{k \neq l} t_1^{h_k} t_2^{h_l} \rangle \tag{29}$$

or rather its connected part:

$$K_2(t_1, t_2) \equiv U_2(t_1, t_2) - U_1(t_1) U_1(t_2) \tag{30}$$

A similar calculation yields

$$U_2(t_1,t_2) = \frac{1}{N^2}\sum_{k,l}\frac{(t_1\alpha_k - t_2\alpha_l)(\alpha_k-\alpha_l)}{(t_1\alpha_k-\alpha_l)(\alpha_k-t_2\alpha_l)} \prod_{m\neq k}\frac{(\alpha_m - t_1\alpha_k)}{(\alpha_m-\alpha_k)}\prod_{m\neq l}\frac{(\alpha_m-t_2\alpha_l)}{(\alpha_m-\alpha_l)} \prod_k\left(\frac{1-\alpha_k}{1-\alpha_k t_1}\right)^N\left(\frac{1-\alpha_k}{1-\alpha_k t_2}\right)^N, \tag{31}$$

from which one derives again a contour integral representation over two complex variables (similar to the representation for the hermitian matrix model found in [17]):

$$K_2(t_1,t_2) = -\frac{1}{N^2}\oint\oint\frac{dudv}{(2i\pi)^2(u-t_2v)(v-t_1u)}\left(\frac{1-u}{1-t_1u}\right)^N\left(\frac{1-v}{1-t_2v}\right)^N \prod_j\frac{(t_1u-\alpha_j)(t_2v-\alpha_j)}{(u-\alpha_j)(v-\alpha_j)} \tag{32}$$

Finally, for the density correlator

$$\rho(p,q) = \langle\sum_k\delta_{p,h_k}\sum_m\delta_{q,h_m}\rangle = \oint\oint\frac{dt_1}{2i\pi\ t_1^{p+1}}\frac{dt_2}{2i\pi\ t_2^{q+1}}K_2(t_1,t_2) \tag{33}$$

we obtain (after changing variables from t_1 to t_1/u and t_2 to t_2/v), a factorized formula, again in similarity with the hermitian matrix model, [17]:

$$\rho(p,q) = -R(p,q)R(q,p) \tag{34}$$

where

$$R(p,q) = \oint\oint\frac{dudtt^{-p}u^{q-p-1}}{(2\pi)^2(1-t)}\left(\frac{u-1}{t-1}\right)^N\prod_j\frac{(t-\alpha_j)}{(u-\alpha_j)} \tag{35}$$

or, going back to the previous non-scaled variables:

$$R(p,q) = -\oint\oint\frac{dudtt^{-p}u^q}{(2\pi)^2(u-t)}\prod_j\frac{(t-\alpha_j)}{(t-1)}\frac{(u-1)}{(u-\alpha_j)} \tag{36}$$

Another useful form of $R(p,q)$ can be obtained (similarly to (21)) by the contour of integration in u in (36) and changing $u\to 1/w$ to catch the poles at $w=0$:

$$R(p,q) = \oint\oint\frac{dwdtt^{-p}u^{-q}}{(2\pi)^2}\prod_j\frac{(\alpha_j-t)}{(1-t)}\frac{(1-w)}{(1-\alpha_jw)}\frac{1}{(1-tw)} \tag{37}$$

Expanding the last factor we get a finite sum representation for the correlator:

$$R(p,q) = \alpha^{-N+q} \sum_{k=0}^{\inf(p,q)} \mathcal{L}_N^{p-k}(\alpha)\hat{\mathcal{L}}_N^{q-k}(\alpha) \tag{38}$$

where $\mathcal{L}_N^q(\alpha)$ and $\hat{\mathcal{L}}_N^q(\alpha)$ are given by (23) and (24).

In the next sections we will study these formulae in the large N limit, for the case of large Young tableaux.

4 Study of the density in the large N limit

In the large N limit the formula for the resolvent (19) gives

$$G(\sigma) = \int_0^\infty \frac{d\tau}{\tau} \oint \frac{du}{2\pi u} \exp\left(-\tau[\sigma - \frac{u}{1-u} - uG_0(u)]\right) \tag{39}$$

where $G_0(u) = \frac{1}{N}\sum_{j=1}^{N} \frac{1}{u-\alpha_j}$ is the resolved of distribution of parameters α_j of our problem. Integration over τ leads to the eq.:

$$G(\sigma) = \oint \frac{du}{2\pi u} \log\left([\sigma - \frac{u}{1-u} - uG_0(u)]\right) \tag{40}$$

Differentiating the last eq. in σ, taking into account the contribution of the pole at

$$\sigma = \frac{u}{1-u} + uG_0(u) \tag{41}$$

and integrating back in σ we obtain an explicit equation for the resolvent

$$G(s) = \ln u(s) \tag{42}$$

where $u(s)$ is a solution of the eq.(41). The constant of integration in (42) is chosen to be zero in order to match the assymptotics $\sigma \to \infty$ $(u \to 1)$.

If we eliminate u from the eqs. (41) and (42) we obtain the following functional equation for $G(\sigma)$

$$e^{G(\sigma)} = 1 + \frac{1}{\sigma} + \frac{1}{\sigma}G_0(e^{G(\sigma)}) \tag{43}$$

This functional formula is very similar to those found by the direct analysis of the saddle point equations in [4, 5] for the heat kernel on the group $SU(N)$ in the large N limit (see also [6, 7] for many other similar results).

For the particular case $\alpha_1 = \alpha_2 = \cdots = \alpha_N = \alpha$ we have $G_0(u) = \frac{1}{u-\alpha}$ and $G(\sigma)$ is given by

$$G(\sigma) = \log\left(\frac{1}{2\sigma}\left((\alpha+1)\sigma - (1-\alpha) - (1-\alpha)\sqrt{[\sigma - \frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}}][\sigma - \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}]}\right)\right) \tag{44}$$

Since on the real axis

$$G(\sigma \pm 0) = ReG(\sigma) \pm i\pi\rho(\sigma) \tag{45}$$

we see from (44) that

$$\rho(\sigma) = 1, \quad for \quad 0 < \sigma < b \tag{46}$$

$$\rho(\sigma) = \log\left(\frac{1}{2\sigma}\left((\alpha+1)\sigma - (1-\alpha) - (1-\alpha)\sqrt{(\sigma-b)(a-\sigma)}\right)\right), for \quad b < \sigma < a \tag{47}$$

where

$$a = \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}, \qquad b = \frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}}. \tag{48}$$

This coincides with the direct saddle point calculation of the Appendix and reproduces an old result of A. Vershik and S. Kerov [10] on the limiting shape of the Young tableau.

Another interesting problem corresponds to the statistics of the weights close to the upper or lower end of the distribution [12, 13] etc. studied in the next section.

5 Double scaling limits

The density function contains three special points ("end points"): $z_c = 0, a, b$, near which it exhibits a universal behavior. We will study the close vicinity of the points a, b such that the deviation Δz from one of these points scales with N according to the power law $\Delta z \sim N^{-2/3}$ typical of the double scaling limit for the generic

distributions of the eigenvalues of matrix models [19]. The scaling for $\alpha \to 1$ in the vicinity $z \sim b \sim 0$ will be different and will be considered afterwards.

It is useful to put in (19) $t = e^{\tau/N} = 1 + \theta/N$. This leads to the integral representation:

$$G(z) = \frac{1}{N}\int_0^\infty \frac{d\theta}{\theta(1+\theta/N)^{zN+1}} \oint \frac{du}{2\pi u}\left(1 - \frac{u}{1-u}\frac{\theta}{N}\right)^{-N} \prod_j \left(1 + \frac{u}{u-\alpha_j}\frac{\theta}{N}\right) \quad (49)$$

Exponentiating the integrand and expanding in $1/N$ we get:

$$G(z) = \int_0^\infty \frac{d\theta}{\theta(1+\theta/N)} \oint \frac{du}{2\pi u} \exp\Big[- \theta \left(z - \frac{u}{1-u} - uG_0(u)\right)$$
$$+\frac{\theta^2}{2N}\left(z + \frac{u^2}{(1-u)^2} + u^2 G_0'(u)\right) - +\frac{\theta^3}{3N^2}\left(-z + \frac{u^3}{(1-u)^3} + \frac{1}{2}u^3 G_0''(u)\right) + \cdots\Big] \quad (50)$$

The end points a, b are defined by the equation (41). The singular behavior arises when the linear in θ term in (50) develops a double zero at the the end point (i.e., when its derivative in u is zero):

$$G_0(u) + uG_0'(u) + \frac{1}{(1-u)^2} = 0 \quad (51)$$

One sees immediately that if the conditions (41) and (51) are satisfied the quadratic in θ term is also zero.In the vicinity of the end points $z = a$ or $z = b$ we can expand in the exponential in powers of $\Delta = z - a$ (or respectively $(z - b)$) and $\delta u = u - u_c$:

$$\frac{\partial}{\partial z}G(z) =$$
$$-\int_0^\infty d\theta \oint \frac{du}{2i\pi u} \exp\Big[- \theta \left(\Delta - \frac{1}{2}A(\delta u)^2\right) + \frac{\theta^2}{2N}\left(\Delta + Au_c\delta u\right) + Au_c^2\frac{\theta^3}{6N^2} + \cdots\Big] \quad (52)$$

with $A = -\frac{2z_c}{u_c^2} + \frac{2u_c}{(1-u_c)^3} + u_c G_0''$. In the double scaling regime, when $\Delta \sim N^{-2/3}$, the quadratic term in $\theta^2\Delta/N$ terms is negligible, and after Gaussian integration in u along the contour of the stationary phase, we obtain an Airy-like function of $\Delta N^{2/3}$

$$\frac{\partial}{\partial z}G(z) \simeq \int_0^\infty \frac{d\theta}{\theta^{1/2}} e^{-\Delta N^{2/3}\theta - B\theta^3}. \quad (53)$$

This gives for the density a double scaling expression:

$$\rho(\sigma) \sim \Delta^{1/2} f(\Delta N^{2/3}) \quad (54)$$

where the function $f(x)$ of the double scaling parameter can be defined nonperturbatively by an appropriate change of the integration contour in (53).

For the last end point of the distribution, namely the vicinity of $z = 0$, in the simple large N limit $\rho(z) = 1$ around this point. But it will be not so any more in the special double scaling limit in which α approaches 1 and the interval $(0, b)$, on which ρ remains fixed to one, shrinks to zero. We will consider this special limit in section 7.

6 Study of the density correlator for nearby levels

Now we shall study the quantity $R(p, q)$ in the large N limit for a separation of p, q of order one : $|p - q| = 1, 2, 3, \cdots$. The result is expected to be universal (if we express $p - q$ in terms of the local level spacing $\rho(p)^{-1}$).

The study of the large N limit of $R(p, q)$ is very similar to the study of the density $\rho(p)$ in the previous section. Namely we put again $t = e^{\tau/N}$ and retain in (36) only the terms of the order ~ 1. The integration over τ yields the following large N limit of R:

$$R(p, q) = \frac{[u_+(q/N)]^{|p-q|} - [u_-(q/N)]^{|p-q|}}{|p - q|} \tag{55}$$

where $u_\pm(\sigma) = e^{G(\sigma)}$ are two conjugated solutions of (41) corresponding to two choices of the sign in (45) when approaching the real σ-axis. Hence we get on the real axis:

$$R(p, q) = 2ie^{2ReG(q/N)} \frac{\sin\left(\pi\rho(q/N)|p - q|\right)}{|p - q|} \tag{56}$$

The level correlation function (34) is thus

$$\rho(p, q) \sim \frac{\sin^2\left(\pi\rho(q/N)|p - q|\right)}{|p - q|^2} \tag{57}$$

It reproduces the well-known universal formula for the level correlation function with Dyson repulsion law (for the correlations of eigenvalues of the unitary ensemble of matrices). It does not change even on the saturated part of the corresponding Young

tableau where $\rho(q/N) = 1$. The only difference with respect to the hermitian ensemble is that in our case of discrete levels is that the this correlation function makes sense only at discrete values of the distance between levels $|p - q| = 1, 2, 3, \cdots$.

The authors of [12, 13, 14, 15] came to the same conclusion in a particular case of equal α's. Our generalization to any collection of α_k's shows once again a remarkable universality of the classical result (57).

7 A special large N limit: small weights in very large Young tableaux

In this section we study a new singular scaling regime, in which all the $\alpha_k \to 1$ and $N \to \infty$, so that the parameter $\rho = N(1 - \frac{1}{N}\sum_k \alpha_k)$ remains finite.

In this limit the polynomial $\mathcal{L}_N^{(p)}(\alpha)$ defined by (23) becomes :

$$\mathcal{L}_N^{(p)}(\alpha) \to \oint \frac{dt}{2\pi i t^{p+1}} e^{\rho/(t-1)} = e^{-\rho} \oint \frac{dz(1+z)^{p-1}}{2\pi i z^{p+1}} e^{-\rho z} = e^{-\rho} M_p(\rho), \tag{58}$$

with

$$M_p(\rho) = L_p(\rho) - L_{p-1}(\rho) \tag{59}$$

in which $L_n(\rho)$ is the standard Laguerre polynomial of order n, normalized to $L_n(0) = 1$. Similarly, in this limit,

$$\hat{\mathcal{L}}_N^{(p)}(\alpha) \to M_p(\rho) \tag{60}$$

The formula (22) becomes:

$$\mathcal{P}_\infty(p) = e^{-\rho} \sum_{k=0}^{p} M_k(\rho) M_{k+1}(\rho). \tag{61}$$

In the limit $\rho \to 0$ of ultra large Young tableaux, we have $M_p(\rho) \simeq -\rho$ and the formula (61) gives explicitly

$$\mathcal{P}_\infty(p) = \rho - (p+1)\rho^2 + O(\rho^3) \tag{62}$$

In full analogy with these formulae for the density we can deduce from (38) the correlation function of the small weights p and q for large Young tableaux:

$$\mathcal{R}_\infty(p,q) = \sum_{k=0}^{\inf(p,q)} M_{p-k}(\rho) M_{q-k}(\rho) \tag{63}$$

which gives for the limit $\rho \to 0$ of ultra large Young tableaux

$$\mathcal{R}_\infty(p,q) = -\rho + \inf(p,q)\rho^2 + O(\rho^3) \tag{64}$$

8 Conclusion

In this work we have studied discrete statistics of the kind (2). Such character expansions (CE) appear in many situations in which one sums over the irreducible representations of $GL(N)$: the hermitian matrix models, circular ensembles, in various partition functions and loop averages of gauge theories , such as those encountered in two-dimensional quantum chromodynamics [1, 2]. The systematic large N saddle point analysis of various CE's was first proposed in [3] in relation to the calculation of partition functions and Wilson loops of QCD_2 on the sphere, and developed further in [4, 5, 7]. It appeared to be the only effective method in the study of the combinatorics of the so called dually weighted planar graphs (DWG) [6] first introduced in [8].

Although we studied only a particular example of such models our results suggest that the universal properties of various correlation functions observed for our model will hold for many other models defined as multiple sums over discrete levels (random ensembles on the Young tableaux).

The explicit formulae, exact for finite N, which have been obtained here-above for the density of levels and for the correlation functions, provide a powerful handle on the study of various universal scaling limits.

It would be nice to apply our methods to more complicated characteristics of the model, such as the probability of distribution of the length of various rows of the Young tableau, (which in the case of equal α's shows, according to the authors of

[11], a remarkable relation to the Painleve II equation). Unfortunately, for such a quantity, we have not succeeded to write it in the form of a finite contour integral representation, similar to that used in our paper for the correlation functions.

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9 Appendix : Density of levels by a direct saddle point calculation

It is interesting to compare the result obtained in section 5 with a direct calculation in the large$N$-limit. For the simplest model in which all the α's are equal, the method is straightforward ; it is based, as usual, on the solution of a Riemann-Hilbert problem.

The probability weight of a given set of random h's, is proportional by definition to $\alpha^{\sum h_k}\Delta(h_1, \cdots, h_N)$. Is we write it in terms of the density distribution

$$\rho(x) = \frac{1}{N}\sum_1^N \delta(x - h_k/N), \tag{65}$$

this weight is proportional to
$\exp N^2 \left(\log\alpha \int dx x\rho(x) + \int dx dy \log|x-y|\rho(x)\rho(y)\right)$. The similarity with the unitary ensembles of random matrices, with their characteristic logarithmic repulsion, is manifest. In the large-N limit the weight is maximum for a density which satisfies the condition

$$\log\alpha + 2P\int dy \frac{\rho(y)}{x-y} = 0 \tag{66}$$

. This condition holds on the support of the measure, which is yet to be determined. We shall make the ansatz (proposed in [3] in a similar situation) that $\rho(x)$ remains equal to one for $0 < x < b$, is some function of x in the interval $b < x < a$, vanishes at $x = a$ and remains zero for $x > a$. The consistency of these conditions will be

checked later. The equation for $\rho(x)$ on the interval b, a becomes

$$\log\sqrt{\alpha} + \log\left(\frac{x}{x-b}\right) + P\int_b^a dy \frac{\rho(y)}{x-y} = 0 \tag{67}$$

We introduce the resolvent

$$G(z) = \int_0^a dy \frac{\rho(y)}{z-y} \tag{68}$$

, and

$$H(z) = \int_b^a dy \frac{\rho(y)}{z-y} \tag{69}$$

. The function $H(z)$ is analytic in the cut-plane from b to a. On the cut it satisfies

$$\mathrm{ReH(x)} + \log\frac{\mathrm{x}\sqrt{\alpha}}{\mathrm{x-b}} = 0. \tag{70}$$

One verifies readily that the function

$$H(z) = \frac{-1}{\pi}\sqrt{(z-a)(z-b)}\int_b^a \frac{dx}{z-x}\frac{1}{\sqrt{(x-b)(a-x)}}\log\left(\frac{x\sqrt{\alpha}}{x-b}\right) \tag{71}$$

is the only fuction which satisfies the analycity requirements and the behaviour at infinity. If we consider a large circle in the complex u-plane, the behavior at infinity implies that

$$\oint \frac{du}{z-u}\frac{1}{\sqrt{(u-b)(u-a)}}\log\left(\frac{u\sqrt{\alpha}}{u-b}\right) = 0. \tag{72}$$

Shrinking now the contour, and collecting the various singularities, we obtain

$$H(z) = -\log\left(\frac{z\sqrt{\alpha}}{z-b}\right) - \sqrt{(z-a)(z-b)}\int_0^b \frac{dx}{z-x}\frac{1}{\sqrt{(a-x)(b-x)}}. \tag{73}$$

The integration is then elementary and we end up with

$$G(z) = -\frac{1}{2}\log\alpha - 2\log\frac{\sqrt{b(z-a)} + \sqrt{a(z-b)}}{\sqrt{z(a-b)}}. \tag{74}$$

The behavior at infinity $G(z) \sim \frac{1}{z}$ fixes a and b ; one finds

$$a = \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}} \text{and} \quad \mathrm{b} = \frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}}. \tag{75}$$

One finds then

$$G(z) = \log \frac{1}{2\alpha z}\Big((1+\alpha)z - (1-\alpha) - (1-\alpha)\sqrt{z-a)(z-b)}\Big), \tag{76}$$

in agreement with the result of section 5. The density of levels is then obtained as

$$\rho(x) = -\frac{1}{\pi}\mathrm{ImG(x+i0)}, \tag{77}$$

and one verifies our assumptions : $\rho(x) = 1$ for $0 < \mathrm{x} < \mathrm{b}$, vanishes for $x > a$ and in the interval (b, a)

$$\rho(x) = \frac{2}{\pi}\arctan\sqrt{\frac{b(a-x)}{a(x-b)}}. \tag{78}$$

This result has been first obtained by A. Vershik [10].

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